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ELECTROMAGNETIC RADIATION FROM A VIBRATING, ELASTIC SPHERE

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Abstract—When the contribution of the polarization gradient to the stored energy is taken into account, in the theory of elastic dielectrics, there is a small coupling between electromagnetic and elastic energies even in isotropic materials. It is found that the electromagnetic radiation from an isotropic, elastic, dielectric sphere vibrating in its fundamental rotatory mode, with a maximum shear strain of 10^{-3}, is of the order of 10^{-38} watts: independent of the radius of the sphere and proportional to the square of the strain.

INTRODUCTION

According to the classical theory of the elastic dielectric continuum, there is no coupling between the mechanical displacement and the electronic polarization in centrosymmetric materials. However, if the contribution of the polarization gradient to the stored energy of deformation and polarization is taken into account, in addition to the usual strain and polarization, such a coupling does exist. Even in the material of highest symmetry (centrosymmetric isotropic) an elastic shear wave induces a transverse polarization wave which, in turn, excites an electromagnetic wave. Thus, with suitable boundary conditions, an electromagnetic radiation may be expected to emanate from any dielectric solid vibrating in a mode involving shear. The simplest example, for a finite body, is that of an isotropic, elastic sphere in rotatory vibration—a mode of vibration in which every spherical surface concentric with the boundary rotates back and forth through a small angle about a single axis. The radiated energy can be expected, of course, to be extremely small; but there is some interest in discovering how small. In the case examined here (a sphere with the isotropized material constants of sodium chloride) the radiation rate accompanying the fundamental mode of rotatory vibration, with a maximum shear strain of 10^{-3}, is of the order of 10^{-38} watts: independent of the radius of the sphere and proportional to the square of the strain.

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

For the centrosymmetric isotropic case, the energy density, W^L, of strain, polarization and polarization gradient is [1]

$$2W^L = a_{11}\mathbf{P}\cdot\mathbf{P} + b_{12}\,\nabla\cdot\mathbf{P}\nabla\cdot\mathbf{P} + b_{44}(\nabla\mathbf{P}:\nabla\mathbf{P} + \nabla\mathbf{P}:\mathbf{P}\nabla) + b_{77}(\nabla\mathbf{P}:\nabla\mathbf{P} - \nabla\mathbf{P}:\mathbf{P}\nabla)$$
$$+ c_{12}\,\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{u} + c_{44}(\nabla\mathbf{u}:\nabla\mathbf{u} + \nabla\mathbf{u}:\mathbf{u}\nabla) + 2d_{12}\,\nabla\cdot\mathbf{P}\,\nabla\cdot\mathbf{u} + d_{44}(\nabla\mathbf{P}:\nabla\mathbf{u} + \nabla\mathbf{P}:\mathbf{u}\nabla), \quad (1)$$

where $\mathbf{u}$ is the mechanical displacement, $\mathbf{P}$ is the electronic polarization, ∇ is the gradient operator and $\nabla\cdot$ is the divergence operator. Corresponding to (1), the mechanical and electrical equations of motion are [1]

$$c_{44}\,\nabla^2\mathbf{u} + (c_{12}+c_{44})\,\nabla\nabla\cdot\mathbf{u} + d_{44}\,\nabla^2\mathbf{P} + (d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P} = \rho\ddot{\mathbf{u}} \tag{2}$$

$$d_{44}\,\nabla^2\mathbf{u} + (d_{12}+d_{44})\,\nabla\nabla\cdot\mathbf{u} + (b_{44}+b_{77})\,\nabla^2\mathbf{P} + (b_{12}+b_{44}-b_{77})\,\nabla\nabla\cdot\mathbf{P} - a_{11}\mathbf{P} + \mathbf{E} = 0 \tag{3}$$

where $\mathbf{E}$ is the Maxwell electric self-field, ρ is the mass density and ∇^2 is Laplace's operator To these equations are adjoined the equations of the electromagnetic field [2, p. 75]

$$c^2\nabla^2\mathbf{A} + \varepsilon_0^{-1}\dot{\mathbf{P}} = \ddot{\mathbf{A}}, \qquad c^2\nabla^2\varphi - \varepsilon_0^{-1}c^2\nabla\cdot\mathbf{P} = \ddot{\varphi}, \qquad c^2\nabla\cdot\mathbf{A} = -\dot{\varphi}, \tag{4}$$

where c and ε_0 are the velocity of electromagnetic waves and the dielectric permittivity in a vacuum and $\mathbf{A}$ and φ are the vector and scalar potentials defined in terms of $\mathbf{E}$ and the magnetic flux density, $\mathbf{B}$, through

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}. \tag{5}$$

We shall be concerned only with transverse waves, in which case φ and all divergences vanish. Then (2)–(5) reduce to

$$c_{44}\nabla^2\mathbf{u} + d_{44}\nabla^2\mathbf{P} = \rho\ddot{\mathbf{u}}, \tag{6}$$

$$d_{44}\,\nabla^2\mathbf{u} + (b_{44}+b_{77})\,\nabla^2\mathbf{P} - a_{11}\mathbf{P} - \dot{\mathbf{A}} = 0, \tag{7}$$

$$c^2\,\nabla^2\mathbf{A} + \varepsilon_0^{-1}\dot{\mathbf{P}} = \ddot{\mathbf{A}} \tag{8}$$

in the elastic dielectric and to

$$c^2\,\nabla^2\mathbf{A} = \ddot{\mathbf{A}} \tag{9}$$

in a vacuum.

If the surface S, with outward normal $\mathbf{n}$, separating the dielectric from a vacuum, supports a surface traction $\mathbf{t}$ but is otherwise free, the boundary conditions on S, when $\nabla\cdot\mathbf{u}$ and $\nabla\cdot\mathbf{P}$ are zero, are [1]

$$\mathbf{n}\cdot\partial W^L/\partial\tfrac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla) = \mathbf{n}\cdot[c_{44}(\nabla\mathbf{u}+\mathbf{u}\nabla) + d_{44}(\nabla\mathbf{P}+\mathbf{P}\nabla)] = \mathbf{t}, \tag{10}$$

$$\mathbf{n}\cdot\partial W^L/\partial\nabla\mathbf{P} = \mathbf{n}\cdot[d_{44}(\nabla\mathbf{u}+\mathbf{u}\nabla) + b_{44}(\nabla\mathbf{P}+\mathbf{P}\nabla) + b_{77}(\nabla\mathbf{P}-\mathbf{P}\nabla)] = 0, \tag{11}$$

along with the usual electromagnetic conditions of continuity of

$$\mathbf{n}\times\mathbf{B},\ \mathbf{n}\cdot\mathbf{B},\ \mathbf{n}\times\mathbf{E},\ \mathbf{n}\cdot(\varepsilon_0\mathbf{E}+\mathbf{P}) \tag{12}$$

across S [2, p. 3].

SOLUTION

We consider the rotatory vibrations of an isotropic, elastic dielectric sphere, of radius a, surrounded by a vacuum. Guided by the solution for rotatory vibrations of a purely elastic sphere ([3] p. 285), we take, in spherical coordinates, r, θ, φ, for $r \le a$,

$$u_r = 0,\ u_\theta = 0,\ u_\varphi = C_1\psi(\alpha r)\sin\theta\cos(\omega t+\varepsilon), \tag{13}$$

$$P_r = 0,\ P_\theta = 0,\ P_\varphi = C_2\,\psi(\alpha r)\sin\theta\cos(\omega t+\varepsilon), \tag{14}$$

$$A_r = 0,\ A_\theta = 0,\ A_\varphi = C_3\,\psi(\alpha r)\sin\theta\sin(\omega t+\varepsilon), \tag{15}$$

where

$$\psi(\alpha r) = (\alpha r)^{-1}\cos\alpha r - (\alpha r)^{-2}\sin\alpha r. \tag{16}$$

Then, for example,

$$\nabla^2 \mathbf{u} = \mathbf{e}_\varphi r^{-2}[(r^2\psi')' - 2\psi]C_1 \sin\theta \cos(\omega t + \varepsilon) = - \mathbf{e}_\varphi \alpha^2 u_\varphi, \tag{17}$$

where prime designates differentiation with respect to r and $\mathbf{e}_\varphi$ is a unit vector in the direction of φ increasing. Similar results hold for the Laplacians of $\mathbf{P}$ and $\mathbf{A}$. Accordingly, upon substitution of (13), (14) and (15) in (6), (7) and (8), the latter become

$$(c_{44}\alpha^2 - \rho\omega^2)C_1 + d_{44}\alpha^2 C_2 = 0, \tag{18}$$

$$d_{44}\alpha^2 C_1 + [(b_{44} + b_{77})\alpha^2 + a_{11}]C_2 + \omega C_3 = 0, \tag{19}$$

$$\omega C_2 + \varepsilon_0(c^2\alpha^2 - \omega^2)C_3 = 0, \tag{20}$$

which have a nonzero solution if

$$\Delta \equiv \begin{vmatrix} c_{44}\alpha^2 - \rho\omega^2 & d_{44}\alpha^2 & 0 \\ d_{44}\alpha^2 & (b_{44} + b_{77})\alpha^2 + a_{11} & \omega \\ 0 & \omega & \varepsilon_0(c^2\alpha^2 - \omega^2) \end{vmatrix} = 0 \tag{21}$$

and this is a cubic equation in α^2 with three real roots: two positive and one negative. Thus, the dispersion relation (21) has two real branches and one imaginary branch. The characters of the branches may be identified by examining their behaviors at low frequencies. For the real branches,

$$\lim_{\omega,\alpha\to 0} \Delta = \lim_{\omega,\alpha\to 0} (c_{44}\alpha^2 - \rho\omega^2)(c^2\alpha^2 - K\omega^2)\varepsilon_0 a_{11} = 0, \tag{22}$$

where K is the dielectric constant:

$$K = 1 + (\varepsilon_0 a_{11})^{-1}, \tag{23}$$

since $\varepsilon_0 a_{11}$ is the reciprocal dielectric susceptibility. Hence, at low frequencies, the dispersion relations of the two real branches are

$$\alpha_1^2 = \rho\omega^2/c_{44} \text{ (acoustic branch)} \tag{24}$$

$$\alpha_2^2 = K\omega^2/c^2 \text{ (electromagnetic branch).} \tag{25}$$

For the third branch, we have

$$\lim_{\omega\to 0} \Delta = \varepsilon_0 c^2\alpha^4\{[c_{44}(b_{44} + b_{77}) - d_{44}^2]\alpha^2 + a_{11}c_{44}\} = 0 \tag{26}$$

or

$$\alpha_3^2 = -a_{11}c_{44}/[c_{44}(b_{44} + b_{77}) - d_{44}^2] \quad \text{(surface branch).} \tag{27}$$

Since positive definiteness of W^L requires $c_{44}(b_{44} + b_{77}) - d_{44}^2$, c_{44} and a_{11} to be positive, α_3 is imaginary.

Taking into account the three branches and separating the two phases of the functions u_φ, P_φ and A_φ, we have, as their forms for $r \leq a$,

$$u_\varphi = \sum_{j=1}^{3} (C_{1j} \cos\omega t + C'_{1j} \sin\omega t)\psi(\alpha_j r)\sin\theta, \tag{28}$$

$$P_\varphi = \sum_{j=1}^{3} (C_{2j} \cos\omega t + C'_{2j} \sin\omega t)\psi(\alpha_j r)\sin\theta, \tag{29}$$

$$A_\varphi = \sum_{j=1}^{3} (C_{3j} \sin\omega t + C'_{3j} \cos\omega t\})\psi(\alpha_j r)\sin\theta, \tag{30}$$

where

$$\psi(\alpha_j r) = (\alpha_j r)^{-1} \cos \alpha_j r - (\alpha_j r)^{-2} \sin \alpha_j r, \quad j = 1, 2, \tag{31}$$

$$\psi(\alpha_j r) = (\alpha_j r)^{-1} \cosh \alpha_j r - (\alpha_j r)^{-2} \sinh \alpha_j r, \quad j = 3. \tag{32}$$

Note that, in (32), the sign of α_3 has been reversed from that in (27) so that, in the sequel, α_3 is real.

The eighteen constants C_{ij} and C'_{ij} are subject to twelve relations through equations (18), (19) and (20), from which we may define

$$\gamma_{2j} = \frac{C_{2j}}{C_{1j}} = \frac{C'_{2j}}{C'_{1j}} = \frac{\mp d_{44}\alpha_j^2 \varepsilon_0(\omega^2 \mp c^2\alpha_j^2)}{\varepsilon_0(\omega^2 \mp c^2\alpha_j^2)[a_{11} \pm (b_{44} + b_{77})\alpha_j^2] + \omega^2}, \tag{33}$$

$$\gamma_{3j} = \frac{C_{3j}}{C_{1j}} = \frac{C'_{3j}}{C'_{1j}} = \frac{\mp d_{44}\,\omega\alpha_j^2}{\varepsilon_0(\omega^2 \mp c^2\alpha_j^2)[a_{11} \pm (b_{44} + b_{77})\alpha_j^2] + \omega^2}, \tag{34}$$

where the upper signs are for $j = 1, 2$ and the lower signs are for $j = 3$. Then (28)–(30) may be written in terms of the six remaining constants C_{1j} and C'_{1j}:

$$u_\varphi = \sum_{j=1}^{3} (C_{1j} \cos \omega t + C'_{1j} \sin \omega t)\psi(\alpha_j r) \sin \theta, \tag{35}$$

$$P_\varphi = \sum_{j=1}^{3} (C_{1j} \cos \omega t + C'_{1j} \sin \omega t)\gamma_{2j}\psi(\alpha_j r) \sin \theta, \tag{36}$$

$$A_\varphi = \sum_{j=1}^{3} (C_{1j} \sin \omega t + C'_{1j} \cos \omega t)\gamma_{3j}\psi(\alpha_j r) \sin \theta. \tag{37}$$

In $r \geq a$, we take $\mathbf{u}$, $\mathbf{P}$, A_r, $A_\theta = 0$ and $A_\varphi = A_\varphi^0$, where

$$\begin{aligned} A_\varphi^0 = {} & C_0[(\alpha_0 r)^{-1} \cos \alpha_0(r - a - ct) - (\alpha_0 r)^{-2} \sin \alpha_0(r - a - ct)]\sin \theta \\ & + C'_0[(\alpha_0 r)^{-1} \sin \alpha_0(r - a - ct) + (\alpha_0 r)^{-2} \cos \alpha_0(r - a - ct)]\sin \theta \end{aligned} \tag{38}$$

and

$$\alpha_0 = \omega/c. \tag{39}$$

To maintain the electromagnetic radiation expressed by (38), an external action on the sphere is required. This we take to be a surface traction on $r = a$:

$$\mathbf{t} = \mathbf{e}_\varphi T \sin \theta \cos \omega t. \tag{40}$$

Noting that, in the present case

$$\nabla\mathbf{u} + \mathbf{u}\nabla = (\mathbf{e}_r\mathbf{e}_\varphi + \mathbf{e}_\varphi\mathbf{e}_r)\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + (\mathbf{e}_\theta\mathbf{e}_\varphi + \mathbf{e}_\varphi\mathbf{e}_\theta)\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} - \frac{u_\varphi \cot \theta}{r}\right), \tag{41}$$

with a similar expression for $\nabla\mathbf{P} + \mathbf{P}\nabla$; and also

$$\nabla\mathbf{P} - \mathbf{P}\nabla = (\mathbf{e}_r\mathbf{e}_\varphi + \mathbf{e}_\varphi\mathbf{e}_r)\left(\frac{\partial P_\varphi}{\partial r} + \frac{P_\varphi}{r}\right) + (\mathbf{e}_\theta\mathbf{e}_\varphi + \mathbf{e}_\varphi\mathbf{e}_\theta)\left(\frac{1}{r}\frac{\partial P_\varphi}{\partial \theta} + \frac{P_\varphi \cot \theta}{r}\right) \tag{42}$$

and $\mathbf{n} = \mathbf{e}_r$, we find that the boundary conditions (10) and (11) become

$$\left[c_{44}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + d_{44}\left(\frac{\partial P_\varphi}{\partial r} - \frac{P_\varphi}{r}\right)\right]_{r=a} = T \sin\theta \cos\omega t, \tag{43}$$

$$\left[d_{44}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{\partial r}\right) + b_{44}\left(\frac{\partial P_\varphi}{\partial r} - \frac{P_\varphi}{r}\right) + b_{77}\left(\frac{\partial P_\varphi}{\partial r} + \frac{P_\varphi}{r}\right)\right]_{r=a} = 0. \tag{44}$$

As for the continuity conditions (12), we note that, in the present case,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mathbf{e}_r}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial r}(rA_\varphi), \tag{45}$$

$$\mathbf{E} = -\mathbf{e}_\varphi \mathring{\mathbf{A}}_\varphi, \tag{46}$$

$$\varepsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{e}_\varphi(-\varepsilon_0 \mathring{\mathbf{A}}_\varphi + P_\varphi), \tag{47}$$

whence:

$$\mathbf{n} \times \mathbf{B} = \mathbf{e}_r \times \mathbf{B} = -\mathbf{e}_\varphi(\partial A_\varphi/\partial r + A_\varphi/r), \tag{48}$$

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{e}_r \cdot \mathbf{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\varphi \sin\theta), \tag{49}$$

$$\mathbf{n} \times \mathbf{E} = -\mathbf{e}_r \times \mathring{\mathbf{A}} = \mathbf{e}_\theta \mathring{\mathbf{A}}_\varphi, \tag{50}$$

$$\mathbf{n} \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \mathbf{e}_r \cdot \mathbf{e}_\varphi(-\varepsilon_0 \mathring{\mathbf{A}}_\varphi + P_\varphi) = 0. \tag{51}$$

Hence, the four continuity conditions (12) reduce to

$$[A_\varphi]_{r=a} = [A_\varphi^0]_{r=a}, \tag{52}$$

$$[\partial A_\varphi/\partial r]_{r=a} = [\partial A_\varphi^0/\partial r]_{r=a}. \tag{53}$$

Upon substituting (35)–(38) in the four conditions (43), (44), (52) and (53) and equating coefficients of $\cos\omega t$ and $\sin\omega t$ separately, we find the following eight equations on the eight constants C_{1j}, C'_{1j}, C_0 and C'_0:

$$\sum_{j=1}^{3} C_{1j}\beta_{1j} = T, \qquad \sum_{j=1}^{3} C'_{1j}\beta_{1j} = 0, \tag{54}$$

$$\sum_{j=1}^{3} C_{1j}\beta_{2j} = 0, \qquad \sum_{j=1}^{3} C'_{1j}\beta_{2j} = 0, \tag{55}$$

$$\sum_{j=1}^{3} C_{1j}\beta_{3j} + C_0[2(\alpha_0 a)^{-3} - (\alpha_0 a)^{-1}] - 2C'_0(\alpha_0 a)^{-2} = 0, \tag{56}$$

$$\sum_{j=1}^{3} C'_{1j}\beta_{3j} + 2C_0(\alpha_0 a)^{-2} + C'_0[2(\alpha_0 a)^{-3} - (\alpha_0 a)^{-1}] = 0, \tag{57}$$

$$\sum_{j=1}^{3} C_{1j}\gamma_{3j}\psi_j - C_0(\alpha_0 a)^{-2} + C'_0(\alpha_0 a)^{-1} = 0, \tag{58}$$

$$\sum_{j=1}^{3} C'_{1j}\gamma_{3j}\psi_j - C_0(\alpha_0 a)^{-1} - C'_0(\alpha_0 a)^{-2} = 0, \tag{59}$$

where $\psi_j = \psi_j(\alpha_j a)$ and

$$\beta_{1j} = (c_{44} + d_{44}\gamma_{2j})(\partial\psi_j/\partial a - \psi_j/a), \tag{60}$$

$$\beta_{2j} = (d_{44} + b_{44}\gamma_{2j})(\partial\psi_j/\partial a - \psi_j/a) + b_{77}\gamma_{2j}(\partial\psi_j/\partial a + \psi_j/a), \tag{61}$$

$$\beta_{3j} = \gamma_{3j}\alpha_0^{-1}\,\partial\psi_j/\partial a. \tag{62}$$

The solution of the eight equations (54)–(59) is

$$C_{1j} = (GA_{3j} + A_{1j})T/|\beta_{ij}|, \qquad C'_{1j} = G'A_{3j}T/|\beta_{ij}|, \tag{63}$$

$$C_0 = F_2\{[2(\alpha_0 a)^{-3} - (\alpha_0 a)^{-1}]F_1 + (\alpha_0 a)^{-2}\}T/D, \tag{64}$$

$$C'_0 = -F_2[2(\alpha_0 a)^{-2}F_1 + (\alpha_0 a)^{-1}]T/D, \tag{65}$$

where $|\beta_{ij}|$ is the determinant with elements β_{ij}, the A_{ij} are the cofactors of the β_{ij} in $|\beta_{ij}|$ and

$$D = \{[(\alpha_0 a)^{-1} - 2(\alpha_0 a)^{-3}]F_1 - (\alpha_0 a)^{-2}\}^2 + [2(\alpha_0 a)^{-2}F_1 + (\alpha_0 a)^{-1}]^2 \tag{66}$$

$$|\beta_{1j}|F_1 = \sum_{j=1}^{3}\gamma_{3j}\psi_j A_{3j}, \qquad |\beta_{1j}|F_2 = \sum_{j=1}^{3}\gamma_{3j}\psi_j A_{1j}, \tag{67}$$

$$G = -F_2\{[4 + (\alpha_0 a)^4]F_1 + (\alpha_0 a)^3 + 2(\alpha_0 a)\}/(\alpha_0 a)^6 D, \qquad G' = -F_2/(\alpha_0 a)^2 D. \tag{68}$$

APPLICATION

The only values of the constants b and d that are known, at this point in time, are those for alkali halides determined by Askar *et al.* [4]. We shall use, here, their isotropized values of b_{44}, b_{77}, d_{44} and c_{44} for sodium chloride [5] as corrected subsequently by Lee:

$$\begin{aligned} b_{44} &= 0{\cdot}222 \times 10^4 \text{ dyn cm}^4/\text{C}^2,\\ b_{77} &= 0{\cdot}218 \times 10^4 \text{ dyn cm}^4/\text{C}^2\\ d_{44} &= -1{\cdot}61 \times 10^6 \text{ dyn cm/C},\\ c_{44} &= 1{\cdot}49 \times 10^{11} \text{ dyn/cm}^2. \end{aligned} \tag{69}$$

In addition, we require numerical values of the dielectric constant, K, and mass density, ρ, of sodium chloride, and the permittivity, ε_0, and velocity of electromagnetic waves, c, in a vacuum:

$$\begin{aligned} K &= 5{\cdot}6 \text{ [6, p. 69]},\\ \rho &= 2{\cdot}214 \text{ gm/cm}^3 \text{ [7, p. 88]},\\ \varepsilon_0 &= 8{\cdot}854 \times 10^{-21} \text{ C}^2/\text{dyn cm}^2 \text{[6, p. 68]},\\ c &= 2.998 \times 10^{10} \text{ cm/sec [2, p. 11]}. \end{aligned} \tag{70}$$

We shall calculate the radiation associated with the fundamental mode of rotatory vibration of the sphere. Since the radiation is extremely small, the frequency of that mode is very nearly that of the corresponding mode of the purely elastic sphere. From (54), this frequency is determined by the lowest root of $\beta_{11} = 0$ with $d_{44} = 0$, i.e. [3]

$$\tan\alpha_1 a = 3\alpha_1 a/(3 - \alpha_1^2 a^2) \text{ or } \alpha_1 a = 5{\cdot}763. \tag{71}$$

Since, also, the frequency of the fundamental mode is low in comparison with those of the higher modes, the wave numbers, α_i, are determined very closely by (24), (25) and (27):

$$\alpha_1^2 = \rho\omega^2/c_{44}, \qquad \alpha_2^2 = K\omega^2/c^2, \qquad \alpha_3^2 = a_{11}c_{44}/[c_{44}(b_{44}+b_{77}) - d_{44}^2], \tag{72}$$

noting, again, that the sign of α_3^2 has been reversed after (27). Thus, we have, from (69), (70) and (72),

$$\alpha_2 a = \alpha_1 a(Kc_{44}/\rho)^{1/2}/c = 1{\cdot}180 \times 10^{-4}, \qquad \alpha_3 = 7{\cdot}485 \times 10^7\ \text{cm}^{-1}, \tag{73}$$

and, finally,

$$\alpha_0 a = a\omega/c = \alpha_1 a(c_{44}/\rho)^{1/2}/c = 4{\cdot}987 \times 10^{-5}. \tag{74}$$

The numerical values in (69)–(74) are all that are required to calculate the ψ_j from (31) and (32), the γ_{ij} from (33) and (34), the β_{ij} from (60)–(62) and, finally, the C_{1j}, C'_{1j}, C_0 and C'_0 from (63), (64) and (65) for insertion in the formulas (35)–(38) for u_φ, P_φ, A_φ and A_φ^0 and, thus, to complete the solution.

The rate of energy radiation, per unit area, from a point on the surface of the sphere, is equal to the normal component of the Pointing vector and this must be equal to the rate of working of the surface traction:

$$\begin{aligned}\mathbf{t}\cdot\dot{\mathbf{u}}]_{r=a} &= T\dot{\mathbf{u}}_\varphi]_{r=a}\sin\theta\cos\omega t\\ &= T\omega\sin^2\theta\cos\omega t\sum_{j=1}^{3}(-C_{1j}\sin\omega t + C'_{1j}\cos\omega t)\psi_j.\end{aligned} \tag{75}$$

The average rate over a period $2\pi/\omega$ is

$$(\omega/2\pi)\int_0^{2\pi/\omega}\mathbf{t}\cdot\dot{\mathbf{u}}]_{r=a}\,\mathrm{d}t = \tfrac{1}{2}T\omega\sin^2\theta\sum_{j=1}^{3}C'_{1j}\psi_j, \tag{76}$$

so that the radiation rate from the entire sphere is

$$(\omega/2\pi)\int_0^{2\pi/\omega}\mathrm{d}t\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\mathbf{t}\cdot\dot{\mathbf{u}}]_{r=a}a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \tfrac{4}{3}\pi\omega a^2T\sum_{j=1}^{3}C'_{1j}\psi_j. \tag{77}$$

Now,

$$\omega a = c\alpha_0 a = 1{\cdot}495 \times 10^6\ \text{cm/sec}, \tag{78}$$

and

$$\sum_{j=1}^{3}aC'_{1j}\psi_j = 2{\cdot}394 \times 10^{-47}T\,\text{cm}^2, \tag{79}$$

with T in dynes/cm^2. Hence the rate of radiation from the sphere is

$$1{\cdot}50 \times 10^{-40}T^2\ \text{ergs/sec}. \tag{80}$$

The traction amplitude, T, may be expressed in terms of the maximum shear strain, $(2e_{r\varphi})_{\text{mix}}$, which is located at $\theta = \pi/2$ and $r = a_m$ where a_m is determined by

$$\partial e_{r\varphi}/\partial r = \tfrac{1}{2}\,\partial(\partial u_\varphi/\partial r - u_\varphi/r)/\partial r = 0, \tag{81}$$

in which u_φ for the purely elastic case may be used in view of its preponderant contribution to the strain. From (81) and (28), with $j = 1$, a_m is determined by the lowest root of

$$\tan \alpha_1 a_m = \alpha_1 a_m(9 - \alpha_1^2 a_m^2)/(9 - 4\alpha_1^2 a_m^2) \qquad \text{or} \qquad \alpha_1 a_m = 3{\cdot}342. \tag{82}$$

The amplitude of $(2e_{r\varphi})_{\max}$, at $\theta = \pi/2$, is then,

$$\begin{aligned}(2e_{r\varphi})_{\max} &= a_m^{-1} C_{11}\{[3(\alpha_1 a_m)^{-2} - 1]\sin \alpha_1 a_m - 3(\alpha_1 a_m)^{-1} \cos \alpha_1 a_m\} \\ &= 2{\cdot}708 \times 10^{-8} T.\end{aligned} \tag{83}$$

Hence, for $(2e_{r\varphi})_{\max} = 10^{-3}$,

$$T = 3{\cdot}69 \times 10^4 \text{ dyn/cm}^2. \tag{84}$$

Upon substituting this value of T in (80), we find that, if the maximum shear strain is 10^{-3}, the radiation rate from the sphere, when it is vibrating in its fundamental rotatory mode, is about 2×10^{-31} erg/sec; i.e. of the order of 10^{-38} watts.

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Абстракт — Когда учитывается фактор градиента поляризации в аккумулированной энергии, в теории упругих диэлектриков, тогда оказывается малое сопряжение между энергиями электромагнитной и упругой, даже, в изотропных материалах. Находится, что электромагнитное излучение из изотропной, упругой, диэлектрической сферы, которая колебается по своей основной форме вращения с максимальной деформацией сдвига порядка 10^{-3}, является порядка 10^{-38} ватт; оно не зависит от радиуса сферы и пропорционально к квадрату деформации.